

Chiral Gauge Anomalies on Noncommutative Minkowski Space-time ¹

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Chiral gauge anomalies on noncommutative Minkowski Space-time are computed and have their origin elucidated. The consistent form and the covariant form of the anomaly are obtained. Both Fujikawa's method and Feynman diagram techniques are used to carry out the calculations.

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1 Introduction

The study of chiral anomalies in field theories over commutative manifolds is a well-established subject whose importance both in Physics and Mathematics cannot be overstated –see ref. [1] and references therein. The concept of Noncommutative manifold [2, 3] and the ideas and tools furnished by field theories defined on them may prove decisive [2] in the quest of understanding the fundamental properties of Nature –see ref. [4] and references therein. It is therefore no futile endeavor to study the quantum properties of field theories over noncommutative manifolds which have chiral symmetries at the classical level. We shall be modest in this article and restrict our analysis to field theories over noncommutative Minkowski Space-time. We shall study the quantization of chiral fermions coupled to a classical background $U(N)$ gauge field when the space-time manifold is noncommutative Minkowski. We refer the reader to ref. [5] for a quick introduction to noncommutative manifolds and, in particular, to noncommutative space-time.

This article is divided into two main parts. In the first part we shall use path integral methods –i.e. variants of Fujikawa’s method– to compute the chiral gauge anomaly. We shall obtain thus the consistent form and the covariant form of the gauge anomaly. We shall also see that in both cases the gauge anomaly occurs because of the lack of chiral gauge invariance of the fermionic measure. In the second part we shall understand the UV and IR origin of the chiral anomaly by using diagrammatic techniques. In this second part we shall compute the noncommutative counterpart of the famous triangle anomaly for the gauge currents. Before we start off, let us warn the reader that no global anomaly will be considered in the sequel.

2 The gauge anomaly in the path integral formalism

Let us take a classical chiral, say, right handed, fermion, $\psi_R^j = P_+ \psi^j$, ($P_+ = \frac{1+\gamma_5}{2}$), carrying the fundamental representation of the group $U(N)$. The physics of this fermion interacting with a $U(N)$ classical gauge field on noncommutative Minkowski space-time is given by the classical action

$$S = \int d^4x \bar{\psi}_i \star (i \not{\partial} \psi^i + A_{\mu j}^i \star \gamma^\mu P_+ \psi^j), \quad (2.1)$$

where ψ^j is a Dirac fermion carrying the fundamental representation of $U(N)$ and the complex matrix $A_{\mu j}^i$, with $(A_{\mu j}^i)^* = A_{\mu i}^j$, is the $U(N)$ gauge field. The indices i, j run from 1 to

N . The previous action is invariant under the following chiral gauge transformations:

$$\begin{aligned} (\delta_\omega A_\mu)^i_j &= \partial_\mu \omega^i_j - i A_{\mu k}^i \star \omega^k_j + i \omega^i_k \star A_{\mu j}^k, \\ (\delta_\omega \psi)^i &= i \omega^i_j \star P_+ \psi^j \quad \text{and} \quad (\delta_\omega \bar{\psi})_k = -i \bar{\psi}_k \star \omega^k_i P_-, \end{aligned} \quad (2.2)$$

where $P_- = \frac{1-\gamma_5}{2}$. The complex functions $\omega^i_j = \omega^{*j}_i$, $i, j = 1, \dots, N$, are the infinitesimal gauge transformation parameters. The symbol \star denotes the Moyal product of functions on Minkowski space-time. The Moyal product is defined, in terms of Fourier transforms, thus

$$(f \star g)(x) = \int \frac{d^4 p}{(2\pi)^4} \int \frac{d^4 q}{(2\pi)^4} e^{i(p+q)_\mu x^\mu} e^{-\frac{i}{2} \theta^{\mu\nu} p_\mu q_\nu} \hat{f}(p) \hat{g}(q).$$

Here, $\theta^{\mu\nu}$ is an antisymmetric real matrix either of magnetic type or light-like type. It is for these choices of matrix θ that it is likely that a unitary theory exists at the quantum level, this theory being connected by Wick rotation with its counterpart in noncommutative Euclidean space [6].

The quantum theory in a background classical gauge field is given by $W[A]$, this functional being formally defined as follows

$$e^{-W[A]} \equiv \int d\psi d\bar{\psi} e^{-S[A]} \equiv Z[A]. \quad (2.3)$$

To render the path integral above as well-defined as possible we are setting our theory on non-commutative space-time with Euclidean signature. $S[A]$ is therefore the Euclidean counterpart of the action in eq. (2.1):

$$S = \int d^4 x \bar{\psi}_i i \hat{D}(A)^i_j \psi^j. \quad (2.4)$$

The operator $i \hat{D}(A)$, which acts on Dirac spinors ψ^j as follows

$$i \hat{D}(A)^i_j \psi^j = i \not{\partial} \psi^i + A_{\mu j}^i \star \gamma^\mu P_+ \psi^j, \quad (2.5)$$

is not an hermitian operator, but it has a well-defined eigenvalue problem since it is an elliptic operator. Actually, the principal symbol of this operator is the same as that of its counterpart in ordinary (commutative) space. Note that we take $\gamma^\mu = (\gamma^\mu)^\dagger$ and $\gamma_5 = -\gamma^1 \gamma^2 \gamma^3 \gamma^4$.

Now, for the theory to make sense at the quantum level, we must demand that $W[A]$ in eq. (2.3) be invariant under the gauge transformations in eq. (2.2), i.e.

$$\delta_\omega W = 0 \iff \int d^4 x \omega^i_j(x) (D_\mu[A] \mathcal{J}_\mu)^j_i(x) = 0. \quad (2.6)$$

Here, $D_\mu[A]^i{}_j(\cdot) = \partial_\mu \delta^i{}_j(\cdot) - i A_{\mu j}^i \star (\cdot) + i (\cdot) \star A_{\mu j}^i$ is the covariant derivative. The current $\mathcal{J}_{\mu j}^i$ is formally defined as follows

$$\mathcal{J}_{\mu j}^i(x) \equiv \langle (\psi_\beta^j \star \bar{\psi}_{\alpha i})(x) (\gamma_\mu P_+)_{\alpha\beta} \rangle = \frac{\delta W[A]}{\delta A_{\mu i}^j(x)}, \quad (2.7)$$

where

$$\langle \dots \rangle = \frac{\int d\psi d\bar{\psi} \dots e^{-S[A]}}{\int d\psi d\bar{\psi} e^{-S[A]}}. \quad (2.8)$$

We shall see below that eq. (2.6) does not hold for the quantum theory in spite of the fact that the classical theory is gauge invariant: the gauge symmetry is anomalous at the quantum level. To make correct statements on the properties of $W[A]$, one must properly define first the formal path integral in eq. (2.3). Since the fermion fields are Grassman fields and the classical action in eq.(2.4) is quadratic in those fields, one aims to define, upon renormalization, $Z[A]$ as the determinant of a certain operator with eigenvalues, say, $\{\lambda_n\}_n$:

$$Z[A] = \left[\prod_n \lambda_n \right]_{\text{ren}}.$$

As in ordinary (commutative) Euclidean space there are two conspicuous choices of $\{\lambda_n\}_n$, namely:

- a) $\{\lambda_n\}_n$ is the set of eigenvalues of $i\hat{D}(A)$.
- b) $\{\lambda_n\}_n$ is the set of eigenvalues of $\sqrt{\left(i\hat{D}(A)\right)^\dagger i\hat{D}(A)}$.

The operator $i\hat{D}(A)$ has been defined in eq.(2.5) and $\left(i\hat{D}(A)\right)^\dagger$ is its hermitian conjugate. We shall show shortly that choices a) and b) lead, respectively, to the consistent form and the covariant form of the anomaly. The consistent form –unlike the covariant form– satisfies the noncommutative Wess-Zumino condition [7] and the covariant form transforms covariantly under gauge transformations. The consistent form does not transform covariantly under gauge transformations. As in the ordinary (commutative) Euclidean space case, for choice b), the current $\mathcal{J}_{\mu j}^i(x) = \langle (\psi_\beta^j \star \bar{\psi}_{\alpha i})(x) (\gamma_\mu P_+)_{\alpha\beta} \rangle$ cannot be obtained, upon renormalization, by functional differentiation of the renormalized $W[A]$. Hence, the Quantum Action Principle does not hold in the theory constructed with choice b)(see ref. [8], and references therein, for background information on the Quantum Action Principle). The reader may notice that the results we have just stated are already true if Euclidean space were commutative (see ref. [9] for case a) and ref. [10] for case b), see also ref. [11]).

2.1 The consistent form of the anomaly

We have seen above that $i\hat{D}(A)$, defined in eq.(2.5), is an elliptic operator. Let us denote the set of its eigenvalues by $\{\lambda_n\}_n$. Then, following ref. [9], we introduce two basis of eigenspinors, say, $\{\phi_n\}_n$ and $\{\chi_n\}_n$ defined as follows

$$i\hat{D}(A)\phi_n = \lambda_n \phi_n, \quad \left(i\hat{D}(A)\right)^\dagger \chi_n = \lambda_n^* \chi_n, \quad (2.9)$$

where λ_n^* is the complex conjugate of λ_n . We shall also normalize the eigenspinors so that the following equation holds

$$\int d^4x \chi_n^\dagger(x) \phi_m(x) = \delta_{nm}. \quad (2.10)$$

Now, the Grassman spinor fields $\psi(x)$ and $\bar{\psi}(x)$ can be expanded thus

$$\psi(x) = \sum_n a_n \phi_n(x), \quad \bar{\psi}(x) = \sum_n \bar{b}_n \chi_n^\dagger(x). \quad (2.11)$$

The coefficients $\{a_n\}_n$ and $\{\bar{b}_n\}_n$ being Grassman numbers.

To define the path integral in eq. (2.3) we shall define its measure first. We define

$$d\psi d\bar{\psi} \equiv \prod_n da_n d\bar{b}_n. \quad (2.12)$$

Taking into account eq. (2.4) and eqs. (2.9) – (2.11), one readily shows that

$$S = \sum_n \lambda_n \bar{b}_n a_n.$$

Furnished with this equation and eq. (2.12) we are ready to define the path integral in eq. (2.3) as follows

$$\int d\psi d\bar{\psi} e^{-S[A]} = \int \prod_n da_n d\bar{b}_n e^{-\sum_n \lambda_n \bar{b}_n a_n}. \quad (2.13)$$

One can now work out the Grassman integrations to obtain

$$Z[A] = \prod_n \lambda_n = \text{“det } i\hat{D}(A)\text{”}.$$

Of course, the product of eigenvalues above is ill-defined and needs regularization. One can use Pauli-Villars (PV) regularization, or zeta function regularization, to define a regularized $Z[A]$ and then renormalized it:

$$Z[A]_{\text{PV regularized}} = \prod_n \frac{\lambda_n}{\lambda_n + M} \xrightarrow{\text{renormalization}} Z[A] = (\det i\hat{D}(A))_{\text{renormalized}}. \quad (2.14)$$

We shall not delve more deeply into the renormalization process since we will not need it to obtain the anomaly. Notice that $Z[A]$ in eq. (2.14) cannot be gauge invariant since, under the infinitesimal gauge transformation, g , in eq. (2.2), the operator $i\hat{D}(A)$ does not undergo a similarity transformation; rather, it changes as follows

$$(1 - i\omega P_-) i\hat{D}(A^g) (1 + i\omega P_+) = i\hat{D}(A) + O(\omega^2).$$

However, as also happens in the commutative space-time setting, the modulus of the quantity $(\det i\hat{D}(A))_{\text{renormalized}}$ can be defined in a gauge invariant gauge, and hence, it is the *phase* of $(\det i\hat{D}(A))_{\text{renormalized}}$ which carries the anomaly. Indeed,

$$|\det i\hat{D}(A)|^2 = \det (i\partial_+ i\partial_-) (\det i\mathcal{D}(A))^2,$$

and the Dirac operator $i\mathcal{D}(A)$ can be regularized in a gauge invariant way by using, for instance, the Pauli-Villars method. Note that $\partial_\pm = \partial P_\pm$ and $i\mathcal{D}(A) = i\partial + \mathbb{A}$.

We now turn to the computation of the consistent form of the anomaly. Let us introduce the following definitions

$$\begin{aligned} Z[A + \delta_\omega A] &= \int d\psi' d\bar{\psi}' e^{-S[A + \delta_\omega A]} = \int \prod_n da'_n d\bar{b}'_n e^{-\sum_n \lambda_n [A + \delta_\omega A] \bar{b}'_n a'_n}, \\ Z[A] &= \int d\psi d\bar{\psi} e^{-S[A]} = \int \prod_n da_n d\bar{b}_n e^{-\sum_n \lambda_n [A] \bar{b}_n a_n}, \end{aligned} \quad (2.15)$$

where

$$\begin{aligned} \psi' &= \psi + \delta_\omega \psi = \sum_n a'_n \phi_n, & \bar{\psi}' &= \bar{\psi} + \delta_\omega \bar{\psi} = \sum_n \bar{b}'_n \chi_n^\dagger, \\ \psi &= \sum_n a_n \phi_n, & \bar{\psi} &= \sum_n \bar{b}_n \chi_n^\dagger, \end{aligned}$$

and $\delta_\omega A$, $\delta_\omega \psi$ and $\delta_\omega \bar{\psi}$ are given in eq. (2.2), and ϕ_n and χ_n are defined in eq. (2.9). Next, let us define δJ as follows:

$$\delta J \equiv -i \int d^4x \sum_n \{ \chi_n^\dagger \star \omega \star P_+ \phi_n - \chi_n^\dagger \star \omega \star P_- \phi_n \} = -i \int d^4x \sum_n \{ \chi_n^\dagger \star \omega \star \gamma_5 \phi_n \}. \quad (2.16)$$

Then, it is not difficult to show that, under the gauge transformations in eq. (2.2), the measure of the path integrals in eq. (2.15) changes thus

$$\prod_n da'_n d\bar{b}'_n - \prod_n da_n d\bar{b}_n = \delta J \prod_n da_n d\bar{b}_n + O(\omega^2). \quad (2.17)$$

Finally, taking into account that $\delta_\omega Z[A] = Z[A + \delta A] - Z[A] + O(\omega^2)$ and eqs. (2.15) – (2.17), one obtains that

$$\delta_\omega W[A] = - \int d^4x \omega_j^i(x) (D_\mu[A] \mathcal{J}_\mu^{\text{consistent}})^j_i(x) = -i \int d^4x \sum_n \{ \chi_n^\dagger \star \omega \star \gamma_5 \phi_n \}. \quad (2.18)$$

The current, $\mathcal{J}_\mu^{\text{consistent}}$, is the current, \mathcal{J}_μ , in eq. (2.7), when the path integrals in eq. (2.8) are defined as in eq. (2.13). The current has been labeled “consistent” since as we shall see, when properly defined, its gauge divergence satisfies the Wess-Zumino consistency condition. The right hand side of eq. (2.18) needs regularization. To regulate it we shall use the Gaussian cut-off function furnished by $\{\lambda_n^2\}_n$:

$$\int d^4x \omega_j^i(x) (D_\mu[A] \mathcal{J}_\mu^{\text{consistent}})^j_i(x) = i \lim_{\Lambda \rightarrow \infty} \int d^4x \sum_n \{\chi_n^\dagger \star \omega \star \gamma_5 e^{-\frac{\lambda_n^2}{\Lambda^2}} \phi_n\}. \quad (2.19)$$

By changing to a plane wave basis, the right hand side of eq. (2.19) can be recast thus

$$i \lim_{\Lambda \rightarrow \infty} \int d^4x \omega_j^i(x) \text{tr} \{\gamma_5 (e^{\frac{\not{D}^2}{\Lambda^2}} e^{ipx})^j_i \star e^{-ipx}\}. \quad (2.20)$$

Here, “tr” denotes trace over the Dirac spinor indices. After a long computation [12], eq. (2.20) leads to

$$\int d^4x \omega_j^i(x) (D_\mu[A] \mathcal{J}_\mu^{\text{consistent}})^j_i(x) = \mathcal{A}^{\text{consistent}}(A, \omega),$$

with

$$\mathcal{A}^{\text{consistent}}(A, \omega) = -\frac{i}{24\pi^2} \text{Tr} \int d^4x \varepsilon_{\mu_1\mu_2\mu_3\mu_4} \omega \partial_{\mu_1} [A_{\mu_2} \star \partial_{\mu_3} A_{\mu_4} - \frac{i}{2} A_{\mu_2} \star A_{\mu_3} \star A_{\mu_4}]. \quad (2.21)$$

The symbol “Tr” denotes the trace over the $U(N)$ generators. The right hand side of this equation is called the consistent form of the gauge anomaly since it satisfies the Wess-Zumino consistency condition, which reads

$$\delta_{\omega_1} \mathcal{A}(A, \omega_2) - \delta_{\omega_2} \mathcal{A}(A, \omega_1) = -i \mathcal{A}(A, [\omega_1, \omega_2]). \quad (2.22)$$

That the Wess-Zumino condition holds is a consequence of the fact that

$$(\delta_{\omega_1} \delta_{\omega_2} - \delta_{\omega_2} \delta_{\omega_1}) W[A] = -i \delta_{[\omega_1, \omega_2]} W[A],$$

and that the consistent current satisfies the following equation

$$(\mathcal{J}_\mu^{\text{consistent}})^i_j(x) = \frac{\delta W[A]}{A_{\mu i}^j(x)}.$$

The reader should note that eq. (2.21) leads to the conclusion that the triangle anomaly cancellation condition reads now

$$\text{Tr}(T^a T^b T^c) = 0,$$

whereas its ordinary (for commutative space-time) is $\text{Tr}(\text{T}^a \{ \text{T}^b \text{T}^c \}) = 0$. The reader is referred to refs. [12, 13] for further comments on this new anomaly cancellation condition. The reader should also note that the anomaly arises because the Jacobian that gives the change of the fermionic measure –see eq. (2.17)– fails to be trivial when properly regularized. The reader is referred to ref. [14] for the computation of supersymmetric version of the consistent anomaly above.

One final technical comment. To obtain eq. (2.21) from eq. (2.20) we have dropped an UV divergent contribution which has the form $\delta_\omega \mathcal{C}(A, \Lambda)$. This contribution can be absorbed into a renormalization of $W[A]$ since $\mathcal{C}(A_\mu, \Lambda)$ is a polynomial in the \star product of A_μ and its derivatives.

2.2 The covariant form of the anomaly

The covariant form of the chiral gauge anomaly is obtained by defining the path integral in eq. (2.3) as follows

$$\int d\psi d\bar{\psi} e^{-S[A]} = \int \prod_n da_n d\bar{b}_n e^{-\sum_n \lambda_n \bar{b}_n a_n}, \quad (2.23)$$

the real numbers λ_n , which can be chosen so that $\lambda_n \geq 0$, and the Grassman numbers a_n and \bar{b}_n are defined thus

$$\begin{aligned} \left(i\hat{D}(A)\right)^\dagger i\hat{D}(A)\varphi_n &= \lambda_n^2 \varphi_n, \quad i\hat{D}(A)\left(i\hat{D}(A)\right)^\dagger \phi_n = \lambda_n^2 \phi_n, \\ \phi_n &= \frac{1}{\lambda_n} i\hat{D}(A)\varphi_n, \quad \text{if } \lambda_n \neq 0, \quad \text{and } i\hat{D}(A)\varphi_n = 0, \quad \text{if } \lambda_n = 0, \\ \varphi_n &= \frac{1}{\lambda_n} \left(i\hat{D}(A)\right)^\dagger \phi_n, \quad \text{if } \lambda_n \neq 0, \quad \text{and } \left(i\hat{D}(A)\right)^\dagger \phi_n = 0, \quad \text{if } \lambda_n = 0, \\ \int d^4x \varphi_n^\dagger(x) \varphi_m(x) &= \delta_{nm}, \quad \int d^4x \phi_n^\dagger(x) \phi_m(x) = \delta_{nm}, \\ \psi(x) &= \sum_n a_n \varphi_n(x), \quad \bar{\psi}(x) = \sum_n \bar{b}_n \phi_n^\dagger(x). \end{aligned} \quad (2.24)$$

Upon Grassman integration, eqs. (2.3) and (2.23) lead to

$$Z[A] = \prod_n \lambda_n = \text{“det } \sqrt{\left(i\hat{D}(A)\right)^\dagger i\hat{D}(A)} \text{”}. \quad (2.25)$$

Unlike in the consistent anomaly case, $Z[A]$ is formally gauge invariant under the infinitesimal gauge transformation, call it g , of A_μ in eq. (2.2). Indeed, $(1 - i\omega P_+) \left(i\hat{D}(A^g)\right)^\dagger i\hat{D}(A^g) (1 +$

$i\omega P_+) = \left(i\hat{D}(A)\right)^\dagger i\hat{D}(A) + O(\omega^2)$, so that $\lambda_n(A^g) = \lambda_n(A) + O(\omega^2)$. Pauli-Villars regularization, or zeta function regularization, renders rigorous the statements above on the gauge invariance of the regularized $Z[A]$. We thus conclude, as it is the case for commutative space-time, that $\delta_\omega W[A] = 0$ for $W[A] = -\ln Z[A]$, with $Z[A]$ as defined by the renormalized counterpart of eq. (2.25). Hence, this $W[A]$ carries no gauge anomaly. Unfortunately, the Quantum Action Principle [8] does not hold for the field theory defined by it. Indeed, now

$$\frac{\delta W[A]}{\delta A_{\mu i}^j(x)} \neq \langle (\psi_\beta^i \star \bar{\psi}_{\alpha j})(x) (\gamma_\mu P_+)_{\alpha\beta} \rangle \equiv (\mathcal{J}_\mu^{\text{covariant}})^i_j(x), \quad (2.26)$$

since it can be shown [15] that the covariant current $\mathcal{J}_\mu^{\text{covariant}}$ is not covariantly conserved; i.e.

$$\int d^4x \, \omega_j^i(x) (D_\mu[A] \mathcal{J}_\mu^{\text{covariant}})^j_i(x) = \mathcal{A}^{\text{covariant}}(A, \omega),$$

with

$$\mathcal{A}^{\text{covariant}}(A, \omega) = \frac{i}{32\pi^2} \text{Tr} \int d^4x \, \omega(x) \varepsilon_{\mu_1\mu_2\mu_3\mu_4} F_{\mu_1\mu_2} \star F_{\mu_3\mu_4}(x). \quad (2.27)$$

“Tr” stands for the trace over the $U(N)$ generators. Let us remark that $\mathcal{J}_\mu^{\text{covariant}}$ in eq. (2.26) is defined so that $\langle \dots \rangle$ is given by eq. (2.8) with the path integrals defined as in eqs. (2.23) – (2.25). It can be shown [15] that $\mathcal{A}^{\text{covariant}}(A, \omega)$ comes from the nontrivial regularized Jacobian that yields the change of the fermionic measure under the chiral gauge transformations of eq. (2.2). Here, the regularized Jacobian is defined by using a regularization function constructed with the eigenvalues λ_n^2 in eq. (2.24).

eq. (2.27) is the noncommutative counterpart of the ordinary result obtained in [10]. It is clear that $\mathcal{A}^{\text{covariant}}(A, \omega)$ transforms covariantly under gauge transformations of A_μ and that it does not satisfies the Wess-Zumino consistency condition in eq. (2.22). It is not difficult to show by explicit computation that the consistent anomaly can be turn into the covariant anomaly by the following redefinition of the currents:

$$(\mathcal{J}_\mu^{\text{covariant}})^i_j = (\mathcal{J}_\mu^{\text{consistent}})^i_j + \mathcal{X}_{\mu j}^i,$$

with

$$\mathcal{X}_{\mu j}^i = \frac{i}{48\pi^2} \varepsilon_{\mu\nu\rho\sigma} (A_\nu \star F_{\rho\sigma} + F_{\rho\sigma} \star A_\nu + i A_\nu \star A_\rho \star A_\sigma)^i_j.$$

3 The gauge anomaly and the triangle Feynman diagrams

The contribution to the consistent anomaly equation –the Euclidean version of this equation is in eq. (2.21)– that comes from the three point contribution –the famous triangle diagrams– to $W[A]$ reads thus

$$p_3^{\mu_3} W_{\mu_1 \mu_2 \mu_3}^{a_1 a_2 a_3}(p_1, p_2)^{\text{eps}} = -\frac{1}{24\pi^2} \varepsilon_{\mu_1 \mu_2 \alpha \beta} p_1^\alpha p_2^\beta \left(\text{Tr} \{T^{a_1}, T^{a_2}\} T^{a_3} \cos \frac{1}{2} \theta(p_1, p_2) - i \text{Tr} [T^{a_1}, T^{a_2}] T^{a_3} \sin \frac{1}{2} \theta(p_1, p_2) \right). \quad (3.1)$$

$W_{\mu_1 \mu_2 \mu_3}^{a_1 a_2 a_3}(p_1, p_2)^{\text{eps}}$ denotes the three point contribution to $W[A]$ that involves the Levi-Civita pseudotensor. Notice that now we are back in noncommutative Minkowski space-time. As in ordinary Minkowski space, one can understand the occurrence of the gauge anomaly either as an UV effect or as an IR phenomenon, by evaluating in terms of triangle diagrams the left hand side of eq. (3.1). We shall do this in what remains of the current section; further details can be found in ref. [17].

Chiral anomalies have been computed using Feynman diagrams in refs [16] as well.

3.1 The UV origin of the gauge anomaly

In dimensional regularization –see [17]– the left hand side of eq. (3.1) reads

$$p_3^{\mu_3} W_{\mu_1 \mu_2 \mu_3}^{a_1 a_2 a_3}(p_1, p_2)^{\text{eps}} = e^{-\frac{i}{2} \theta(p_1, p_2)} \text{Tr} T^{a_1} T^{a_2} T^{a_3} \Delta_{\mu_1 \mu_2}(p_1, p_2; d) + e^{\frac{i}{2} \theta(p_1, p_2)} \text{Tr} T^{a_2} T^{a_1} T^{a_3} \Delta_{\mu_2 \mu_1}(p_2, p_1; d),$$

where

$$\Delta_{\mu_1 \mu_2}(p_1, p_2; d) = - \int \frac{d^d q}{(2\pi)^d} \frac{\text{tr}^{\text{eps}} \{ (\not{q} + \not{p}_1) \bar{\gamma}_{\mu_1} P_+ \not{q} \bar{\gamma}_{\mu_2} P_+ (\not{q} - \not{p}_2) (\bar{\not{p}}_1 + \bar{\not{p}}_2) P_+ \}}{(q^2 + i0^+) ((q + p_1)^2 + i0^+) ((q - p_2)^2 + i0^+)}.$$

The previous Feynman integral, which is the ordinary one, yields upon integration an UV pole at $d - 4$. This UV pole is canceled by the contribution $(d - 4) \varepsilon_{\mu_1 \mu_2 \alpha \beta} p_1^\alpha p_2^\beta$ coming from the trace over the gamma matrices. eq. (3.1) is thus obtained along with the UV interpretation of it. Of course, this is the same interpretation as in ordinary Minkowski space-time; an interpretation which can be found in ref. [18].

3.2 The IR origin of the gauge anomaly

To interpret the nonabelian chiral anomaly on noncommutative Minkowski as an IR phenomenon, we shall follow Coleman and Grossman [19] and compute $W_{\mu_1\mu_2\mu_3}^{a_1a_2a_3}(p_1, p_2)^{\text{eps}}$ at the point

$$p_1^2 = p_2^2 = p_3^2 = -Q^2, \quad p_1 + p_2 + p_3 = 0.$$

Some integration [17] yields

$$\begin{aligned} W_{\mu_1\mu_2\mu_3}^{a_1a_2a_3}(p_1, p_2)^{\text{eps}} = \\ \frac{1}{24\pi^2} \left(\frac{1}{Q^2} \right) \left(\text{Tr} \{T^{a_1}, T^{a_2}\} T^{a_3} \cos \frac{1}{2}\theta(p_1, p_2) - i \text{Tr} [T^{a_1}, T^{a_2}] T^{a_3} \sin \frac{1}{2}\theta(p_1, p_2) \right) \\ \left(\varepsilon_{\mu_1\mu_2\alpha\beta} p_1^\alpha p_2^\beta p_3^\mu + \varepsilon_{\mu_3\mu_1\alpha\beta} p_3^\alpha p_1^\beta p_2^\mu + \varepsilon_{\mu_2\mu_3\alpha\beta} p_2^\alpha p_3^\beta p_1^\mu \right). \end{aligned} \quad (3.2)$$

Hence, we conclude that $W_{\mu_1\mu_2\mu_3}^{a_1a_2a_3}(p_1, p_2; p_1^2 = p_2^2 = p_3^2 = -Q^2)^{\text{eps}}$ has an IR singularity at $Q^2 = 0$. The contraction of the right hand side of eq. (3.2) with p_3^μ erases the singularity at $Q^2 = 0$ and yields the triangle anomaly given in eq. (3.1). An explanation of the IR origin of the gauge anomaly has been obtained thus.

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